An Uncertainty Analysis of Some Real Functions for Image Processing Applications

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Abstract

There are many benefits to be gained in image processing and compression by the use of analyzing functions which are local in both space and spatial frequency. It is often assumed that these benefits are in some way proportional to the degree of joint locality of the functions being used. Within the limits imposed by the uncertainty principle, there can be great variation in this joint locality across different local function families. While there is no generally accepted joint locality metric appropriate for visual applications, Gabor's joint uncertainty is often cited to justify the use of a set of functions. It has been shown that complex Gabor functions optimize this metric. There is some debate however, regarding which, of the restricted class of real functions, has the lowest joint uncertainty. In this paper we examine three families of real functions and directly evaluate the Gabor metric for joint uncertainty. In contrast to previous attempts to prove the optimality of any one function, this analysis provides an explicit numerical basis for comparison of these real functions.

1 Introduction

In this paper we examine and compare the joint spatial/spatial frequency locality of three families of real functions which have been used or suggested for a variety of image processing applications. Such local functions are used for filtering and as basis functions in transforms to provide information about image frequency content at specific points within the image.

The degree to which a function is simultaneously local in both space and frequency is limited by the uncertainty principle, which plays an important role in image processing[7]. Gabor has formalized this principle by measuring the uncertainty of a signal in each domain as the variance of the signal power normalized by the signal energy[2]. In 1D, the product of the spatial and spectral uncertainty then has a lower bound of 0.5. The now famous Gabor functions are complex exponential functions in a Gaussian envelope which Gabor showed achieve this lower bound. In large part due to this locality property and the finding that these functions can be used to model the receptive field profiles of simple cells in the visual cortex of cats[5], Gabor functions have found widespread interest in the image processing community.

One disadvantage of using Gabor functions for image processing applications is the fact that they are complex valued functions (acting on image data which is generally real). As is the case with the Fourier transform, this results in the manipulation and storage of complex numbers. For the sake of computational efficiency, real local functions are of interest.

The three real functions examined in this paper are the Hermite function, the Gabor cosine, and the Gaussian derivative. Hermite functions are considered because there are conflicting claims in the literature regarding their optimality. Researchers have suggested that the Hermite functions are the real functions which minimize Gabor's uncertainty product[2, 6] while others claim that these functions maximize the uncertainty[4]. When real functions are required, the real or imaginary parts of the complex Gabor function are commonly used although the uncertainty of these have not been shown to be optimal. Finally, the Gaussian derivatives are jointly local functions that have been identified by Young[9] as appropriate models of receptive field profiles and have been used by the authors as image transform basis functions[1]. There is some disagreement as to whether the findings regarding the optimality of Hermite functions[6] can be extended to Gaussian derivative functions[8].

Absent a conclusive proof that, of the real functions, there is one that has a lower joint uncertainty than all others,

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a direct evaluation of Gabor's uncertainty product for these functions offers a grounds for comparison. We address the problem in 1D. Clearly, the relative comparisons hold in the 2D separable case as well.

2 Joint Uncertainty

The joint uncertainty product as defined by Gabor[2] is

$$U = (\Delta x)(\Delta \omega) \tag{1}$$

where $(\Delta x)^2$ is the effective width and $(\Delta \omega)^2$ is the effective bandwidth of the function of interest. For a real, zero centered function $g(x)$, these widths are

$$(\Delta x)^2 = \frac{\int_{-\infty}^{\infty} x^2 g^2(x) dx}{\int_{-\infty}^{\infty} g^2(x) dx} \tag{2}$$

$$(\Delta \omega)^2 = \frac{\int_{-\infty}^{\infty} \omega^2 G^*(\omega) G(\omega) d\omega}{\int_{-\infty}^{\infty} G^*(\omega) G(\omega) d\omega}. \tag{3}$$

where $G(\omega)$ is the Fourier transform of $g(x)$.

3 The Functions

The three functions for which we will evaluate the joint uncertainty product are listed below. The subscripts HF, GC, and DG denote the Hermite, Gabor cosine, and derivative of Gaussian functions respectively.

$$g_{GC}(x) = \cos(\eta x) e^{-\frac{x^2}{2\sigma^2}} \tag{4}$$

$$g_{DG}(x) = \frac{d^n}{dx^n}\left[e^{-\frac{x^2}{2\sigma^2}}\right] \tag{5}$$

$$= H_{n,\sigma}(x) e^{-\frac{x^2}{2\sigma^2}} \tag{6}$$

$$g_{HF}(x) = H_{n,\sigma}(x) e^{-\frac{x^2}{4\sigma^2}}. \tag{7}$$

In these definitions σ is the standard deviation of a Gaussian, η is the frequency of the cosine, and $H_{n,\sigma}(x)$ is a generalized Hermite polynomial satisfying the following relationships:

$$H_{n,\sigma}(x) = \left(\frac{1}{\sigma\sqrt{2}}\right)^n H_n\left(\frac{x}{\sigma\sqrt{2}}\right) \tag{8}$$

$$H_n(x) = H_{n,\sigma}(x)|_{\sigma=1/\sqrt{2}} \tag{9}$$

where $H_n(x)$ is an n^{th} order Hermite polynomial. The Fourier transforms of the three functions are given by:

$$G_{GC}(\omega) = \frac{\sigma}{2}\left[e^{\frac{1}{2}\sigma^2(\omega-\eta)^2} + e^{\frac{1}{2}\sigma^2(\omega+\eta)^2}\right] \tag{10}$$

$$G_{DG}(\omega) = (j\omega)^n \sigma e^{-\frac{\omega^2\sigma^2}{2}} \tag{11}$$

$$G_{HF}(\omega) = j^n e^{-\sigma^2\omega^2} H_n(\sigma\sqrt{2}\omega)\left(\frac{1}{\sigma\sqrt{2}}\right)^{n-1}. \tag{12}$$

4 Results

The squared joint uncertainty products of the three families of functions described above are found by substitution of the function definitions into the effective width and bandwidth equations (2) and (3). In these derivations, integrals which are said to be known can be found in a table of integrals such as [3]. It is useful to recognize that the denominators of both the effective width and the effective bandwidth are simply the signal energy. This energy will be evaluated in one domain and the result used in both.

4.1 Hermite Functions

The effective width of a Hermite function can be written as:

$$(\Delta x)^2 = \frac{\left(\frac{1}{\sigma\sqrt{2}}\right)^{2n-3} \int_{-\infty}^{\infty} z^2 H_n^2(z) e^{-z^2} dz}{\left(\frac{1}{\sigma\sqrt{2}}\right)^{2n+1} \int_{-\infty}^{\infty} H_n^2(z) e^{-z^2} dz} \tag{13}$$

where $z = \frac{x}{\sigma\sqrt{2}}$ in both the numerator and the denominator. The integral in the denominator is known and the integral of the numerator can be expanded into two,

$$\int_{-\infty}^{\infty} z^2 H_n^2(z) e^{-z^2} dz = \frac{1}{4}\int_{-\infty}^{\infty} (4z^2-2) H_n^2(z) e^{-z^2} dz + \frac{1}{2}\int_{-\infty}^{\infty} H_n^2(z) e^{-z^2} dz, \tag{14}$$

both of which are known since $(4z^2 - 2) = H_2(z)$.

In considering the effective bandwidth we need only evaluate the numerator which can be written as

$$\left(\frac{1}{\sigma\sqrt{2}}\right)^{2n+1} \int_{-\infty}^{\infty} z^2 H_n^2(z) e^{-z^2} dz \tag{15}$$

where we have let $z = \sigma\sqrt{2}\omega$. This integral is the same as that of the numerator of (13) and is known. The squared joint uncertainty of the Hermite functions can now be written as

$$U_{HF}^2 = \left(n + \frac{1}{2}\right)^2. \tag{16}$$

4.2 Gabor Cosine Functions

The effective width of a Gabor cosine can be written as:

$$(\Delta x)^2 = \frac{\int_0^{\infty} x^2 e^{-x^2/\sigma^2}\left(1+\cos(2\eta x)\right) dx}{2\int_0^{\infty} e^{x^2/\sigma^2}\cos^2(\eta x) dx}. \tag{17}$$

The integral of the denominator of (17) is in a known form and that of the numerator can be split into two, each of which is known.

For the effective bandwidth, the Fourier transform of the Gabor function (10) is squared, resulting in the sum of 3 exponentials, and the numerator of (3) becomes

$$\frac{\sigma^2}{4}\int_{-\infty}^{\infty}\omega^2 e^{-\sigma^2(\omega-\eta)^2}d\omega+\frac{\sigma^2}{4}\int_{-\infty}^{\infty}\omega^2 e^{-\sigma^2(\omega+\eta)^2}d\omega+ \quad (18)$$

$$\frac{\sigma^2}{2}\int_{-\infty}^{\infty}\omega^2 e^{-\frac{1}{2}\sigma^2(2\omega^2+2\eta^2)}d\omega. \quad (19)$$

After substituting $\alpha = \omega - \eta$ and $\beta = \omega + \eta$, all three integrals can be expanded and simplified. Those with odd symmetric integrands vanish and those with even symmetric integrands are easily written in known form.

The resulting squared joint uncertainty for the Gabor cosine function is

$$U_{GC}^2 = \frac{1}{4}\left(1+\frac{1-2\rho^2 e^{-\rho^2}-e^{-2\rho^2}}{1\left(1+e^{-\rho^2}\right)}\right), \quad (20)$$

where $\rho^2 = \eta^2\sigma^2$.

4.3 Gaussian Derivative Functions

Since the energy of the Gaussian derivative function is most easily calculated in the Fourier domain, the effective bandwidth will be evaluated first as

$$(\Delta\omega)^2 = \frac{\int_{-\infty}^{\infty}\omega^2\left(\omega^{2n}\sigma^2 e^{-\sigma^2\omega^2}\right)d\omega}{\sigma^2\int_{-\infty}^{\infty}\omega^{2n}e^{-\omega^2\sigma^2}d\omega}. \quad (21)$$

The integral of the denominator is known and that of the numerator is easily modified to a known form.

In order to evaluate the numerator of (2) for the Gaussian derivative, we recognize that

$$\mathcal{F}\{-jxg_n(x)\} = \frac{d}{d\omega}G_n(\omega) \quad (22)$$

and Parseval's relation allows the numerator to be written as

$$\int_{-\infty}^{\infty}\left[\frac{d}{d\omega}G_n(\omega)\right]^*\left[\frac{d}{d\omega}G_n(\omega)\right]d\omega. \quad (23)$$

After differentiation of $G_n(\omega)$ with respect to ω, the numerator,

$$\sigma^2\int_{-\infty}^{\infty}\omega^{2(n-1)}\left(n^2-2n\sigma^2\omega^2+\sigma^4\omega^4\right)e^{\sigma^2\omega^2}d\omega, \quad (24)$$

can be expanded into three integrals each of which is in a known form.

The sqaured joint uncertainty of the Gaussian derivative function can now be written,

$$U_{DG}^2 = \frac{(2n+1)(4n-1)}{4(2n-1)}. \quad (25)$$

n	η	Gaussian Derivative	Gabor Cosine
0	0	0.5	0.5
2	0.471405	1.70783	0.769168
4	0.666667	2.19578	1.37675
6	0.816497	2.60681	1.7738
8	0.942809	2.96367	2.05569
10	1.05409	3.28273	2.2902
12	1.1547	3.57376	2.49981
14	1.24722	3.84298	2.69255
16	1.33333	4.09465	2.87228

Table 1. Comparison of the joint uncertainties of various Gaussian derivatives and Gabor cosines

5 Comparison

While the joint uncertainty does not vary among the members of the complex Gabor function family, it does so for the real function families examined here. This is due, in part, to the fact that the effective bandwidth (3) is calculated about a mean frequency which is zero for real functions. The calculated effective bandwidth of real functions will thus increase as the spectral peak moves higher in frequency. Because the joint uncertainty of these real functions will vary, it is necessary to decide which members of each family to compare. The Hermite functions and the Gaussian derivatives are both products of polynomials and Gaussians. As a result, it is reasonable to compare functions which have the same Gaussian variance and the same polynomial order. By examining U_{HF} and U_{DG} in (16) and (25), it can be seen that the joint uncertainties of these functions are independent of σ. Furthermore, while the uncertainty of the Hermite function increases with n, the Gaussian derivative uncertainty increases with $\sqrt{n}$. $U_{HF} = U_{DG}$ for $n = 0, 1$ and $U_{HF} > U_{DG}$ thereafter.

When considering the Gabor cosine functions, selecting the same Gaussian variance is appropriate. The Gabor cosine center frequency (η) must also be specified. We choose η so that the passbands of the Gaussian derivative (for a specific n) and the Gabor cosine align. The Hermite functions, however, cannot be so aligned because their spectra are themselves Hermite functions and are in general not bandpass, with the exception of the two lowest order functions. For this reason, a direct comparison is made only between the Gaussian derivatives and the Gabor cosines. Table 1 shows the comparison for the first 9 even symmetric Gaussian derivative functions and corresponding Gabor cosines.

6 Conclusion

Analytic expressions for the joint uncertainty have been presented for three real function families that are of interest for image processing applications. Hermite functions are found to have significantly higher joint uncertainty product than Gaussian derivatives for a fixed variance and polynomial order higher than 1. This is contrary to any suggestion of optimality for the Hermite functions with respect do this uncertainty measure. Hermite functions are also poor frequency selectors due to the shapes of their spectra. Gabor cosine functions are found to offer significant improvement in uncertainty product over the Gaussian derivatives.

It was indicated in Section 5 that in the effective bandwidth calculation of (3), $\Delta\omega$ does not correspond to the bandwidth of the function relative to some mean, rather it measures the bandwidth relative to zero. For this reason the resulting metric (1), which has been examined in this paper and is widely used in image processing, may not be the most appropriate measure of joint uncertainty for real functions. The formulation of more appropriate metrics remains a topic of research.

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Errata

Equation 20 should read

$$U_{GC}^2 = \frac{1}{4}\left(1 + \frac{1 - 2\rho^2 e^{-\rho^2} - e^{-2\rho^2}}{\frac{1}{2\rho^2}\left(1 + e^{-\rho^2}\right)^2}\right). \tag{20}$$

The data in Table 1 is correct and was generated using a Gaussian standard deviation of $\sigma = 3.0$.